HAFTA 5 (LİNEER DENKLEM SİSTEMLERİ)

$[A:B]$ arttırılmış matris eşolon forma indirgenir ve aşağıdaki durumlar söz konusudur;

i) $\text{rank}(A) \neq \text{rank}(A:B) \Rightarrow$ sistemin çözümü yoktur.

ii) $\text{rank}(A) = \text{rank}(A:B) \Rightarrow$ sistemin tek çözümü vardır.

a) $r=n$ ise sistemin tek bir çözümü var

b) $r<n$ ise sistemin $(n-r)$ keyfi değişkene bağlı sonsuz tane çözümü vardır.

ÖRNEK: 1
$$\left.\begin{aligned} -2x+y+5z&=1 \\ x+2y-z&=0 \\ 3y+2z&=1 \end{aligned}\right\}$$
lin. denk. sis. çözün.

$$\left[\begin{array}{ccc|c} -2 & 1 & 5 & 1 \\ 1 & 2 & -1 & 0 \\ 0 & 3 & 2 & 1 \end{array}\right] \xrightarrow{\text{eşolon form}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Burada $\text{rank}(A) = \text{rank}(A:B) = 3$ old. sistemin çözümü var $n=3$ ve $r=n$ olduğundan sistemin tek çözümü vardır ve

$x=4$, $y=-1$, $z=2$ dir.

Sistem homojen ise $(AX=0)$ $X=0$ aşikar çözümdür. Buna sıfır çözümde denir. $\text{rank}(A) = \text{rank}(A:B) = r$ olsun

i) $r=n$ ise tek çözüm sıfır çözüm olur

ii) $r<n$ ise sistemin $(n-r)$ keyfi değişkene bağlı sonsuz çözümü vardır.

ÖRNEK : 2

$$\left.\begin{aligned} 3x+2y&=7 \\ 17x+y&=0 \\ 6x+4y&=3 \end{aligned}\right\}$$ lin. denk. sis. çözümü ?

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 7 \\ 17 & 1 & 0 & 0 \\ 6 & 4 & 0 & 3 \end{array}\right] \xrightarrow{\text{eşolon}} \begin{bmatrix} 1 & 0 & -7/31 \\ 0 & 1 & 119/31 \\ 0 & 0 & 1 \end{bmatrix}$$

$\text{rank}(A)=2$, $\text{rank}(A:B)=3$ $\text{rank}(A) \neq \text{rank}(A:B)$ olduğundan lineer denk. sis. çözümü yoktur.

ÖRNEK : 3 $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ve $X=\begin{bmatrix} x \\ y \end{bmatrix}$ olmak üzere $AX=0$ lineer homojen denk. sisteminin ~~katsayılar matrisinin~~ sadece 0 (aşikar) çözümü olması için a, b, c, d arası bağlantı ne olmalı?

$\Rightarrow$ Sadece sıfır çözüm için $AX=0$ lin. hom. denk. sis. katsayılar matrisinin $\text{rank}(A)=\text{rank}(A:B)=n$ olmalıdır. Yani rank = 2 dir. O zaman $|A| \neq 0$ olmalıdır.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \Rightarrow \begin{aligned} &ad-bc \neq 0 \\ &ad \neq bc \text{ olmalı} \end{aligned}$$

ÖRNEK: 4

$$\left.\begin{aligned} 2x-y+4z&=a \\ -x+4y-2z&=0 \\ 3x+2y+6z&=-2 \end{aligned}\right\}$$ lin. denk. sis. çözümünün olması için a ne olmalı?

ÖRNEK 4 $[A:B] = \left[\begin{array}{ccc|c} 2 & -1 & 4 & a \\ -1 & 4 & -2 & 0 \\ 3 & 2 & 6 & -2 \end{array}\right]$

Çözüm:

$$\xrightarrow{R_1+R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & a \\ -1 & 4 & -2 & 0 \\ 3 & 2 & 6 & -2 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & a \\ 0 & 7 & 0 & a \\ 3 & 2 & 6 & -2 \end{array}\right]$$

$$\xrightarrow{R_3-3R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & a \\ 0 & 7 & 0 & a \\ 0 & -7 & 0 & -2-3a \end{array}\right] \xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & a \\ 0 & 7 & 0 & a \\ 0 & 0 & 0 & -2-2a \end{array}\right]$$

$$\xrightarrow{R_2/7 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & a \\ 0 & 1 & 0 & a/7 \\ 0 & 0 & 0 & -2-2a \end{array}\right] \xrightarrow{R_1-3R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4a/7 \\ 0 & 1 & 0 & a/7 \\ 0 & 0 & 0 & -2-2a \end{array}\right]$$

sistemin çözümü olması için $\text{rank}(A) = \text{rank}(A:B) = 2$ olmalı

O halde $-2-2a=0$

$-2a=2$

$\boxed{a=-1}$ olmalıdır.

ÖRNEK: 5

$2x_1 - x_2 = 5$

$x_1 + 3x_2 = 6$ lin. denk. sis. ters matris ile çözün.

$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ olsun.

$A \cdot A^{-1} = I$

$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$2x+y=1$

$-x+3y=0 \rightarrow y=\frac{x}{3}$

$2z+t=0 \rightarrow z=-\frac{t}{2}$

$-z+3t=1$

$2x+\frac{x}{3}=1$ $\qquad$ $\frac{t}{2}+3t=1$

$\boxed{x=\frac{3}{7}}$ $\qquad$ $\boxed{t=\frac{2}{7}}$

$\boxed{y=\frac{1}{7}}$ $\qquad$ $\boxed{z=-\frac{1}{7}}$

$A^{-1} \cdot b = \left(\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right) \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

olur. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ bulunur.

(1) $$\begin{aligned} x+2y+z&=5\\ 2x+2y+z&=6\\ x+2y+3z&=9 \end{aligned}$$ lin. denk. sistemini Cromer yöntemi ile çözünüz.

$$\Delta = |A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -4 \qquad B = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$$

$$\Delta_1 = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4, \quad \Delta_2 = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4, \quad \Delta_3 = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

olduğundan;

$$x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1, \quad y = \frac{\Delta_2}{\Delta} = \frac{-4}{-4} = 1, \quad z = \frac{\Delta_3}{\Delta} = \frac{-8}{-4} = 2$$

(2) $$\begin{aligned} 2x+y+z&=17\\ 4x+2y+2z&=14\\ 3x+y+5z&=2 \end{aligned}$$ lin. denk. sis. çözün.

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 5 \end{vmatrix} = 0$$ old. Cromer ile çözülemez.

$$[A:B] = \begin{bmatrix} 2 & 1 & 1 & 17 \\ 4 & 2 & 2 & 14 \\ 3 & 1 & 5 & 2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -7 & 17 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

olup $\text{rank}(A) = \text{rank}(A:B) = 2$ old. $(3-2)=1$ keyfi değişkene bağlı sonsuz çözüm vardır.

(3) $x_1 - 2x_2 + 5x_3 - 1 = 0$

$2x_1 + x_2 - x_3 = 0$

$3x_1 + 2x_3 - 1 = 0$ lin. denk. sis. Cramer yöntemi ile çözün.

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & 1 & -1 \\ 3 & 0 & 2 \end{vmatrix} = 1 \qquad \Delta_1 = \begin{vmatrix} 1 & -2 & 5 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} = -1$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 0 & -1 \\ 3 & 1 & 2 \end{vmatrix} = 4 \qquad \Delta_3 = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 2$$

$x_1 = \dfrac{\Delta_1}{\Delta} = -1 \qquad x_2 = \dfrac{\Delta_2}{\Delta} = 4 \qquad x_3 = \dfrac{\Delta_3}{\Delta} = 2$ dir.

(4) $A = \begin{bmatrix} 3 & 6 & 9 \\ 1 & 4 & 11 \\ 5 & 14 & 30 \end{bmatrix}$ A matrisinin üçgen ayrışımı varsa bulunuz.

$$[A : I_3] = \left[\begin{array}{ccc|ccc} 3 & 6 & 9 & 1 & 0 & 0 \\ 1 & 4 & 11 & 0 & 1 & 0 \\ 5 & 14 & 30 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1/3 R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/3 & 0 & 0 \\ 1 & 4 & 11 & 0 & 1 & 0 \\ 5 & 14 & 30 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-5R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/3 & 0 & 0 \\ 1 & 4 & 11 & 0 & 1 & 0 \\ 0 & -6 & -25 & 0 & -5 & 1 \end{array}\right] \xrightarrow{-1R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/3 & 0 & 0 \\ 0 & 2 & 8 & -1/3 & 1 & 0 \\ 0 & -6 & -25 & 0 & -5 & 1 \end{array}\right]$$

$$\xrightarrow{3R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/3 & 0 & 0 \\ 0 & 2 & 8 & -1/3 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 1 \end{array}\right] \xrightarrow{1/2 R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/3 & 0 & 0 \\ 0 & 1 & 4 & -1/6 & 1/2 & 0 \\ 0 & 0 & -1 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/3 & 0 & 0 \\ 0 & 1 & 4 & -1/6 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

olur. $[A : I_3] \sim [U \; L^{-1}]$ olduğundan;

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \text{ ve } L^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/6 & 1/2 & 0 \\ 1 & 2 & -1 \end{bmatrix} \text{ dir.}$$

Şimdi L matrisini bulalım;

$$[L^{-1} : I_3] = \left[\begin{array}{ccc|ccc} 1/3 & 0 & 0 & 1 & 0 & 0 \\ -1/6 & 1/2 & 0 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-6R_2 \to R_2]{3R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 0 \\ 1 & -3 & 0 & 0 & -6 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_1 + R_3 \to R_3]{-R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -3 & -6 & 0 \\ 0 & 2 & -1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{-1/3 R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -5 & -4 & 1 \end{array}\right] \xrightarrow{-R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 5 & 4 & -1 \end{array}\right]$$

$$L = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 5 & 4 & -1 \end{bmatrix}$$ olarak bulunur. $A = LU$'dur.